It might help you see this by the following

$$\sqrt{x^2+y^2} = d - x$$

square $\Rightarrow$ $\cancel{x^2} + y^2 = d^2 - 2dx \cancel{+x^2}$

$\Rightarrow$ $2dx = -y^2 + d^2 \Rightarrow x = \frac{-1}{2d} y^2 + \frac{d}{2}$

Lecture 3

Before doing $\varepsilon > 1$ & $\varepsilon < 1$, let's do the following:

$$r = \frac{d}{1+\varepsilon\cos\theta} \Longleftrightarrow r + \varepsilon r\cos\theta = d$$

$$\Rightarrow \pm\sqrt{x^2+y^2} + \varepsilon x = d$$

$$\Rightarrow \pm\sqrt{x^2+y^2} = d - \varepsilon x$$

(square) $\Rightarrow$ $x^2 + y^2 = d^2 - 2\varepsilon dx + \varepsilon^2 x^2$

$$\Rightarrow (1-\varepsilon^2)x^2 + 2\varepsilon dx + y^2 = d^2$$

$$\Rightarrow x^2 + \frac{2\varepsilon d}{1-\varepsilon^2}x + \frac{y^2}{(1-\varepsilon^2)} = \frac{d^2}{1-\varepsilon^2}$$

(Complete the square) $\Rightarrow$ $x^2 + \frac{2\varepsilon d}{1-\varepsilon^2}x + \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} = \frac{d^2}{1-\varepsilon^2} + \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2}$

$$\Rightarrow \left(x + \frac{\varepsilon d}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{d^2}{1-\varepsilon^2}\cdot\frac{\cancel{1-\varepsilon^2}}{1-\varepsilon^2} + \frac{\cancel{\varepsilon^2} d^2}{\cancel{(1-\varepsilon^2)^2}} = \left(\frac{d}{1-\varepsilon^2}\right)^2$$

$$\Rightarrow \boxed{\frac{\left(x + \frac{\varepsilon d}{1-\varepsilon^2}\right)^2}{\left(\frac{d}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{d^2}{1-\varepsilon^2}\right)} = 1} \quad (**)$$

$\boxed{\text{(ii) } \varepsilon < 1}$ this means $1-\varepsilon^2 > 0$

Let $a = \dfrac{d}{1-\varepsilon^2}$, $b = \dfrac{d}{\sqrt{1-\varepsilon^2}}$, and $c = \sqrt{a^2-b^2}$.

$$a^2-b^2 = \frac{d^2}{(1-\varepsilon^2)^2} - \frac{d^2}{1-\varepsilon^2} = \frac{\cancel{d^2} - (\cancel{1}-\varepsilon^2)d^2}{(1-\varepsilon^2)^2} = \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2}$$

So, $c = \sqrt{a^2-b^2} = \dfrac{\varepsilon d}{1-\varepsilon^2} = \varepsilon a$.

(**) now becomes

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

This an ellipse with center $(-c, 0)$, semimajor axis $a = \dfrac{d}{1-\varepsilon^2}$, semiminor axis $b = \dfrac{d}{\sqrt{1-\varepsilon^2}}$, and right focal point at the origin. Because $c = \varepsilon a$, $\varepsilon = \dfrac{c}{a}$ is the <u>eccentricity</u> of the ellipse.

Remember, the foci always sit on semimajor axis, c units away from the center on either side.

Notice that if $\varepsilon < 1$, $1 + \varepsilon \cos\theta > 0$ so $r > 0$.

Notice if $\varepsilon=0$, then

(**) $\dfrac{x^2}{d^2}+\dfrac{y^2}{d^2}=1 \iff x^2+y^2=d^2 \iff r=d$

a circle of radius d w/ center $=(0,0)$.

We can easily see this from (*).

$\boxed{\text{(iii) } \varepsilon>1}$ In this case $1-\varepsilon^2<0$ or $\varepsilon^2-1>0$.

Let $a=\dfrac{d}{\varepsilon^2-1}$, $b=\dfrac{d}{\sqrt{\varepsilon^2-1}}$, and $c=\sqrt{a^2+b^2}$.

$$a^2+b^2=\frac{d^2}{(\varepsilon^2-1)^2}+\frac{d^2}{\varepsilon^2-1}=\frac{\cancel{d^2}+(\varepsilon^2\cancel{-1})d^2}{(\varepsilon^2-1)^2}=\frac{\varepsilon^2 d^2}{(\varepsilon^2-1)^2}=\left(\frac{\varepsilon d}{\varepsilon^2-1}\right)^2$$

$\Rightarrow c=\dfrac{\varepsilon d}{\varepsilon^2-1}=\varepsilon a$. This time we get for (**)

$$\frac{(x-c)^2}{a^2}-\frac{y^2}{b^2}=1$$

Comparing this to $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, we see it is a hyperbola with semimajor axis a, semiminor axis b, asymptotes intersecting at $(c,0)$, and foci at $(0,0)$ and $(2c,0)$.

A word on how hyperbolas graph in polar coordinates

If $\varepsilon > 1$, then $1+\varepsilon\cos\theta$ can be positive, negative, and zero! So, as long as it's defined, r can be positive or negative in this case!

$$(*)\ r = \frac{d}{1+\varepsilon\cos\theta} \iff r + \varepsilon r\cos\theta = d \iff \boxed{r + \varepsilon x = d}$$

(i) $r > 0 \Rightarrow r = \sqrt{x^2+y^2} \geq x$

So, $d = r + \varepsilon x \geq x + \varepsilon x = (1+\varepsilon)x \Rightarrow x \leq \dfrac{d}{1+\varepsilon}$

So, when $r>0$, we get the branch of the hyperbola to the left of the line $x = \dfrac{d}{1+\varepsilon}$.

(ii) $r < 0 \Rightarrow r = -\sqrt{x^2+y^2} \leq -x$

$\Rightarrow d = r + \varepsilon x \leq -x + \varepsilon x = (\varepsilon - 1)x \Rightarrow x \geq \dfrac{d}{\varepsilon - 1}$

So, when $r<0$, we get the branch of the hyperbola to the right of the line $x = \dfrac{d}{\varepsilon-1}$.

Ex: Identify and sketch the following equations:

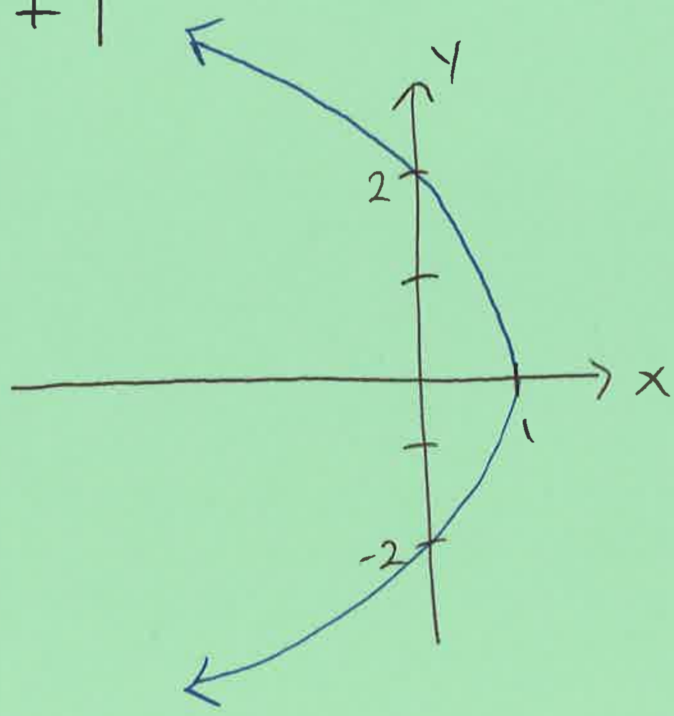
ⓐ $r = \dfrac{2}{1+\cos\theta}$ ⓑ $r = \dfrac{3}{1+\frac{1}{2}\cos\theta}$ ⓒ $r = \dfrac{2}{1+3\cos\theta}$

Sol:

ⓐ $\varepsilon = 1 \Rightarrow$ parabold. $d = 2$ gives

$$x = \frac{-1}{4}y^2 + 1$$

y	0	2	-2
x	1	0	0



ⓑ $\varepsilon = \frac{1}{2} \Rightarrow$ ellipse. $d = 3$ gives

$$a = \frac{3}{1-\frac{1}{4}} = \frac{3}{\frac{3}{4}} = 4, \quad b = \frac{3}{\sqrt{1-\frac{1}{4}}} = \frac{3}{\sqrt{\frac{3}{4}}} = 2\sqrt{3} \approx 3.5$$

$$c = \varepsilon a = \frac{1}{2}\cdot 4 = 2 \longrightarrow \frac{(x+2)^2}{(4)^2} + \frac{y^2}{(2\sqrt{3})^2} = 1$$

See graph on later page

(c) $\varepsilon = 3 \Rightarrow$ hyperbola. Since $d = 2$

$$a = \frac{2}{9-1} = \frac{1}{4}, \quad b = \frac{2}{\sqrt{9-1}} = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}} \approx 0.7$$

$$c = \varepsilon a = \frac{3}{4}.$$

$$\longrightarrow \frac{\left(x - \frac{3}{4}\right)^2}{\left(\frac{1}{4}\right)^2} - \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

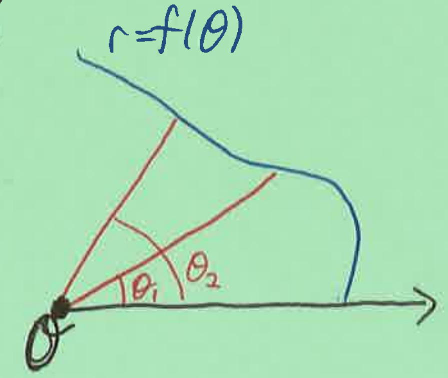
See next page for graph

Derivatives in Polar Coordinates

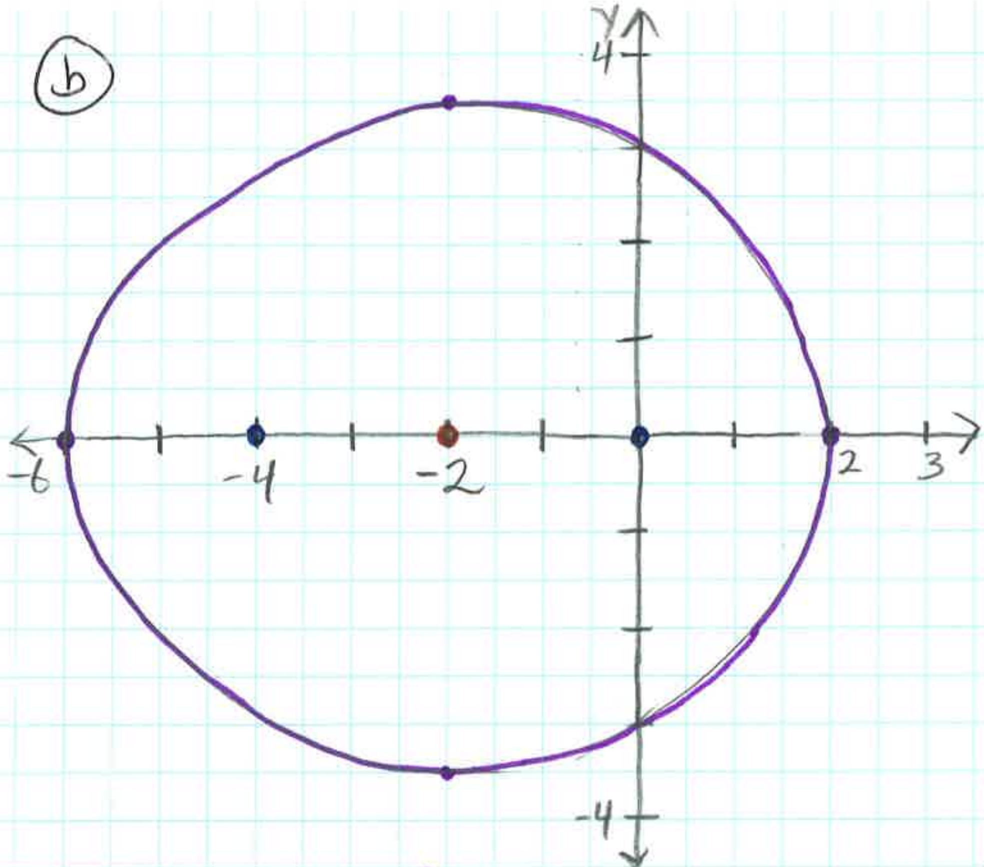
Let $r = f(\theta)$ be a function and $f'(\theta)$ its derivative.

Recall that, in Cartesian coordinates, the derivative gives the slope of the graph. What does the derivative mean in polar coordinates?

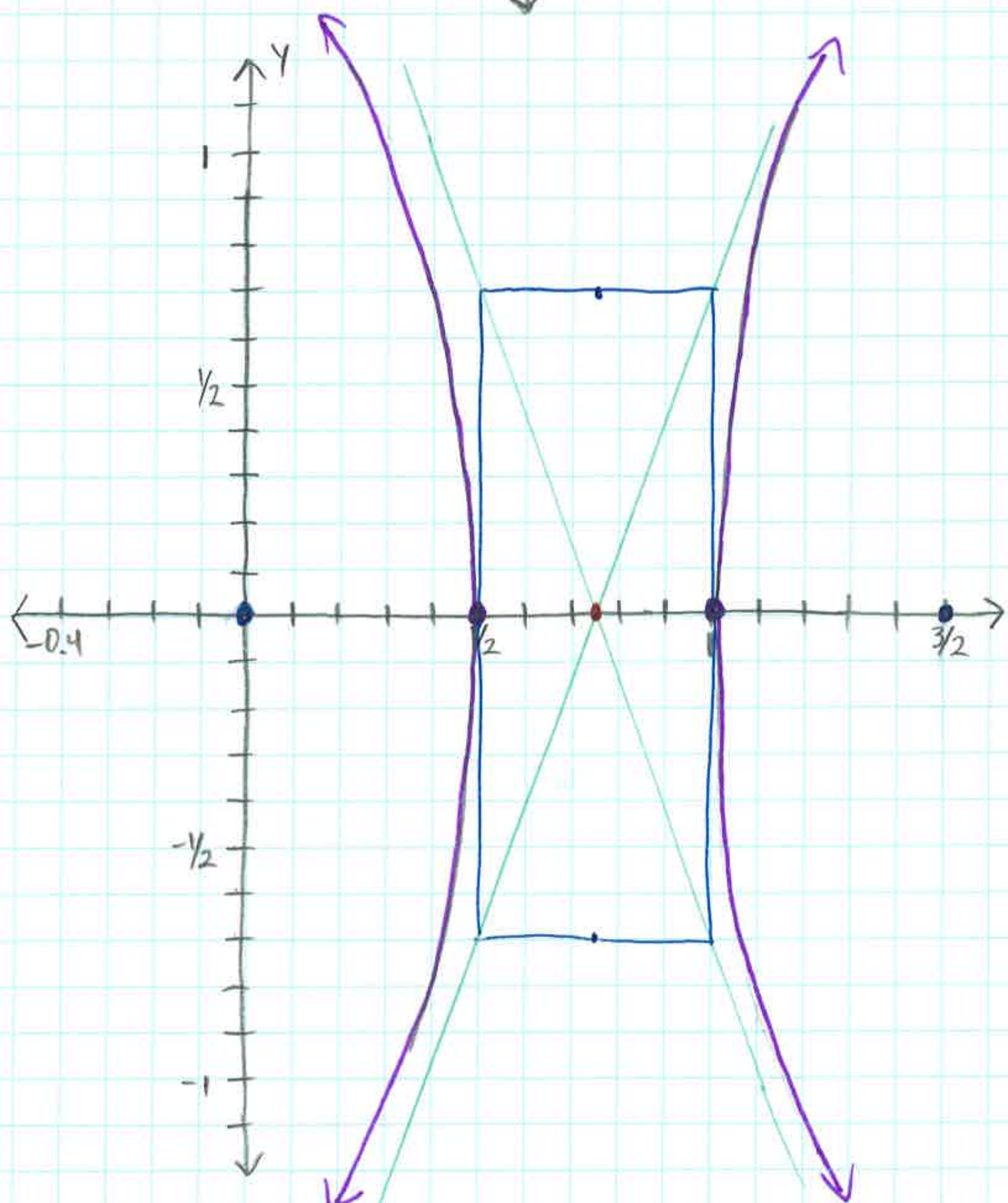
Consider a typical situation:



(b)



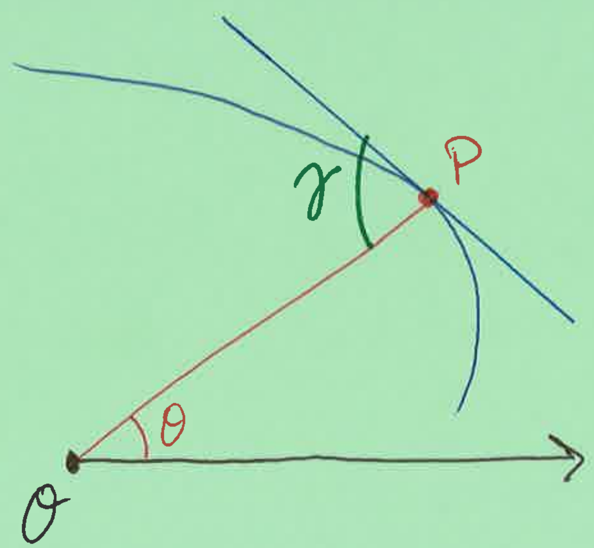
(c)



Suppose $f'(\theta) > 0$ over the interval $\theta_1 \leq \theta \leq \theta_2$. This means that $r = f(\theta)$ is increasing on this interval, i.e., moving further from $\mathcal{O}$.

Likewise, if $f'(\theta) < 0$, then $r = f(\theta)$ is decreasing as θ sweeps from θ_1 to θ_2.

Let's look at this another way now. Let $P = (f(\theta), \theta)$ be a point on the graph of $f(\theta)$ and assume $P \neq \mathcal{O}$, i.e., $f(\theta) \neq 0$. Let $\gamma = \gamma(\theta)$ be the angle, measured counterclockwise, from the tangent line to $f(\theta)$ at P to the line segment $\mathcal{O}P$.



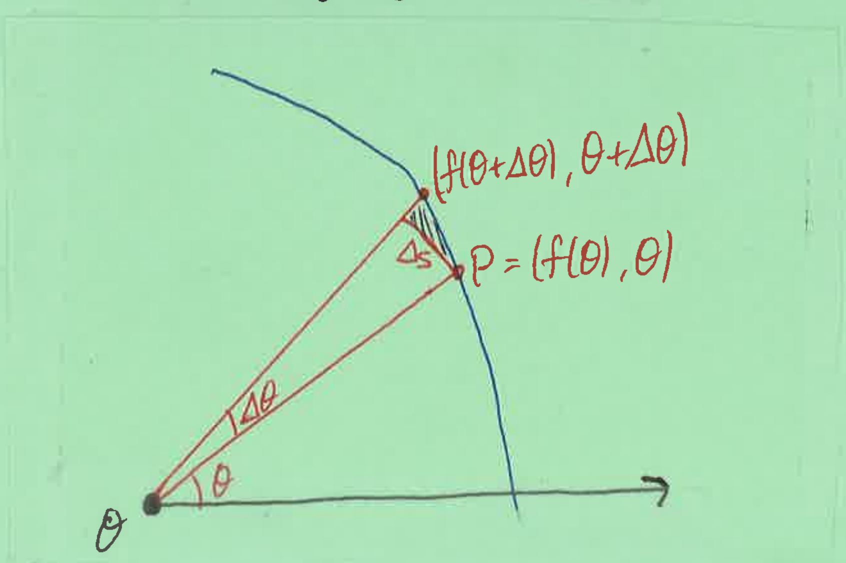
Notice that $0 \leq \gamma < \pi$.

If $\gamma(\theta) > \frac{\pi}{2}$, then we can see that $f(\theta)$ is increasing at θ. If $\gamma(\theta) < \frac{\pi}{2}$, we see that $f(\theta)$ is decreasing at θ. If $\gamma(\theta) = \frac{\pi}{2}$, then $f(\theta)$ isn't changing at θ. Notice that these exactly correspond to the cases $f'(\theta) > 0$, $f'(\theta) < 0$, and $f'(\theta) = 0$, respectively. Thus, it is reasonable to expect a connection between $f'(\theta)$ & $\gamma(\theta)$...

So, what is this connection?

Definition of $f'(\theta)$:

$$f'(\theta) := \lim_{\Delta\theta \to 0} \frac{f(\theta+\Delta\theta) - f(\theta)}{\Delta\theta}$$

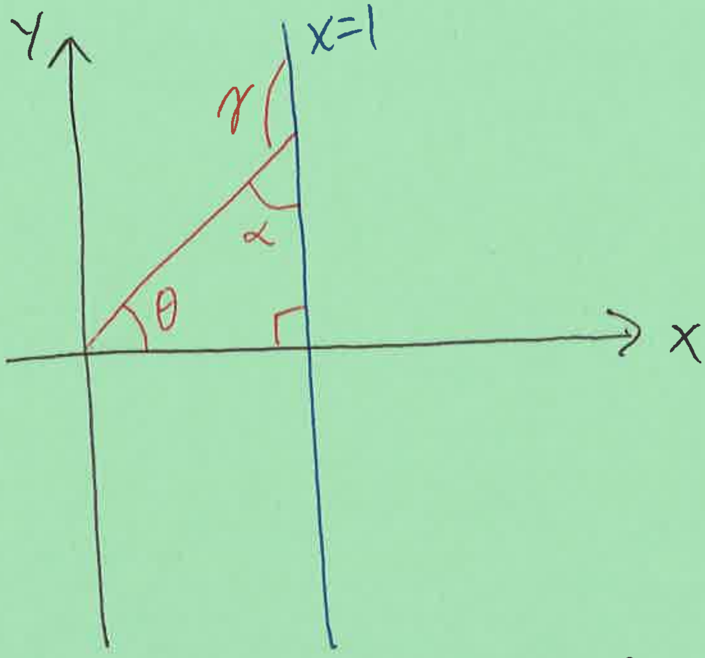


Let's call the (shaded) curved triangle the <u>beak</u> at P.

Recall that $\Delta s = f(\theta)\Delta\theta \Rightarrow \frac{1}{\Delta\theta} = f(\theta)\frac{1}{\Delta s}$.

Ex: Verify $f'(\theta) = f(\theta)\tan\left(\gamma - \frac{\pi}{2}\right)$ for $f(\theta) = \frac{1}{\cos\theta}$

Sol: $r = \frac{1}{\cos\theta} \Longleftrightarrow r\cos\theta = 1 \Longleftrightarrow x = 1$



$$\theta + \alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{2} - \theta$$

$$\gamma + \alpha = \pi$$

$$\Rightarrow \gamma + \frac{\pi}{2} - \theta = \pi$$

$$\Rightarrow \gamma = \theta + \frac{\pi}{2}$$

$$f'(\theta) = \frac{-(-\sin\theta)}{(\cos\theta)^2} = \frac{\tan\theta}{\cos\theta}$$

$$f(\theta)\tan\left(\gamma - \frac{\pi}{2}\right) = \frac{1}{\cos\theta}\tan\left(\theta + \frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{\tan\theta}{\cos\theta} \quad ✓$$